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PERTURBATION METHODS FOR THE SOLUTION OF LINEAR PROBLEMS.(U)
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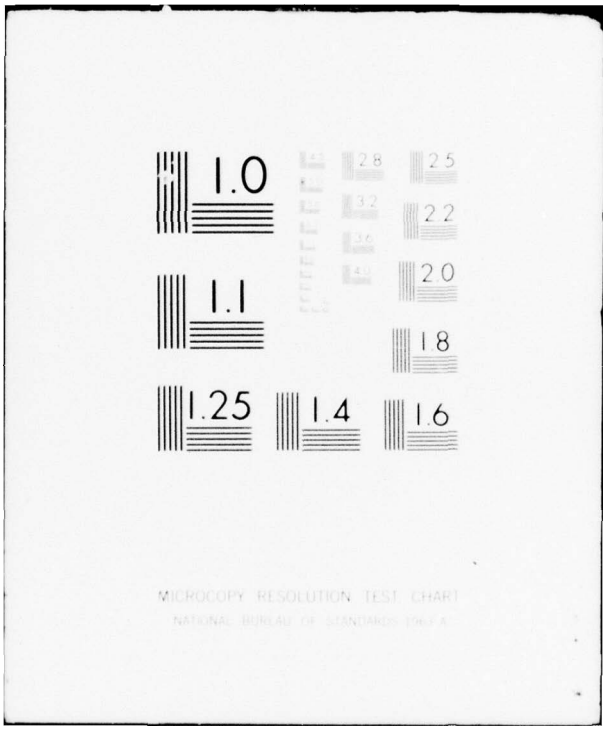
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L. B. Rall

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PERTURBATION METHODS FOR THE SOLUTION OF LINEAR PROBLEMS

L. B. Rall

Mathematics Research Center
University of Wisconsin-Madison

Dedicated to Professor Arvid T. Lonseth on his 65th Birthday

Abstract. Linear problems of central interest in numerical analysis are the solution of linear equations, the construction of the inverse or a generalized inverse of a linear operator, finding the eigenvalues and eigenvectors of a linear operator, and linear programming. A survey is made of methods which apply if the data of a solved linear problem is perturbed by operators and vectors of small norm (analytic perturbation), or by operators of finite rank and vectors belonging to a finite-dimensional subspace (algebraic perturbation). Perturbation methods may be used to extend the theory of linear problems, to estimate errors due to inaccurate data and computation, and to solve perturbed problems with economy of effort.

1. Linear problems. In the abstract framework of functional analysis, a *linear* problem is one which can be formulated in terms of linear spaces and operators [38 , Chapter I]. Naturally, many problems of theoretical and practical interest in numerical analysis belong to this general class. Among these problems, some are important enough to be the subjects of extensive investigations, and also appear in the daily workload of most computing centers devoted to *general scientific computation*. Of these significant problems, the ones singled out for discussion here are: (a) solution of linear equations, (b) inversion of linear operators, (c) finding the eigen-eigenvalues and eigenvectors of a linear operator, and (d) linear programming. These problems will now be defined in appropriate generality.

a. Solution of linear equations.

Let X,Y denote complete normed linear spaces over a common scalar field Λ. In most applications, one has $\Lambda = R$, the real numbers, or $\Lambda = C$, the complex numbers. The notation $L(X,Y)$ will be used for the set of continuous linear operators from X into Y. Given an operator $A \in L(X,Y)$ and a vector $y \in Y$ as *data*, the problem is to find a *solution* $x \in X$ of the *linear equation*

$$Ax = y . \tag{1.1}$$

For practical as well as abstract treatment of this problem, it is important to be in possession of a *theory* of equation (1.1), which provides information as to which of the following alternatives holds:

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(1.1a)
$$\begin{cases} \text{(i)} & \text{For each } y \in Y, \text{ equation (1.1) has a unique solution } x \in X; \\ & \text{or} \\ \text{(ii)} & \text{for some } y \in Y, \text{ equation (1.1) has no solution or several solutions.} \end{cases}$$

The choice between (i) existence and uniqueness or (ii) nonexistence or nonuniqueness or solutions exhausts the logical possibilities, and thus the *alternative structure* (1.1a) will be characteristic of the theory of any equation, linear or nonlinear. In case (i), the operator A is said to be *nonsingular*; otherwise (case (ii)), it is called a *singular* operator.

b. Inversion of linear operators.

This problem is closely related to the solution of linear equations. In the nonsingular case (i), equation (1.1) defines the linear (right) *inverse operator* A^{-1} which gives the unique solution x as

(1.2) $$x = A^{-1}y .$$

In many applications, one has $Y = X$ and A^{-1} satisfies

(1.3) $$A^{-1}A = AA^{-1} = I ,$$

where I denotes the *identity operator* in X, that is, $Ix = x$ for all $x \in X$.

In the singular case (ii), the alternatives are nonexistence or nonuniqueness of solutions x of (1.1). The inverse operator A^{-1} of A does not exist in this case, but one may seek a *generalized inverse* $A^{\dagger}$ of A which has some properties which are desirable for the application at hand. For example, in connection with the problem of solving the linear equation (1.1), one might want

(1.4) $$x = A^{\dagger}y$$

to be a solution if the equation is *consistent*, and thus is satisfied by one or more elements of X. It turns out that this is equivalent to the condition that $A^{\dagger}$ satisfies the operator equation

(1) $$AA^{\dagger}A = A.$$

Any operator $A^{\dagger}$ for which (1) holds will be called an *inner inverse* of A [25, pp. 7-11]. The more formal term *{1}-inverse* of A [3, pp. 7-8] has also been applied specifically to operators $A^{\dagger}$ satisfying condition (1).

If equation (1.1) is consistent and $A^{\dagger}$ is an inner inverse of A, then all solutions x may be represented in the form

(1.5) $$x = A^{\dagger}y + (I - A^{\dagger}A)z$$

for $z \in X$. With z arbitrary, formula (1.5) is called the general solution of equation (1.1), as in the elementary theory of linear differential equations.

Generalized inverses may also be useful in case equation (1.1) has no solutions, and thus is said to be *inconsistent*, or *overdetermined*. Here, the possibility of choosing a *generalized solution* x to minimize the norm of the *residual vector*

$$r = Ax - y \tag{1.6}$$

in Y is considered. Suppose that the set

$$G(A,y) = \{x \mid \|Ax-y\| = \min_{z\in X} \|Az-y\| \} \tag{1.7}$$

of generalized solutions x is nonempty, as will certainly be the case if equation (1.1) is consistent. If Y is a Hilbert space, then elements $x \in G(A,y)$ are ordinarily called *least-squares solutions* of the linear equation (1.1). Thus, one might require that $A^{\dagger}y \in G(A,y)$ for all $y \in Y$ in addition to property (1). Furthermore, the subset

$$S(A,y) = \{x \mid x \in G(A,y),\ \|x\| = \min_{z\in G(A,y)} \|z\| \} \tag{1.8}$$

of $G(A,y)$ may be nonempty, and would then consist of the generalized (or least-squares) solutions x of (1.1) of *minimum norm*. The requirement that $A^{\dagger}y \in S(A,y)$ for all $y \in Y$ would then also be a possible additional restriction on the set of generalized inverses of A. If $S(A,y)$ consists of a single point for each $y \in Y$ then the corresponding generalized inverse $A^{\dagger}$ is uniquely determined. In case X and Y are finite-dimensional Euclidean spaces, this generalized inverse $A^{\dagger}$ exists and is the *Moore-Penrose inverse* of A [3, pp. 7, 103-121], which, in addition to (1), satisfies the condition

$$A^{\dagger}AA^{\dagger} = A^{\dagger}; \tag{2}$$

that is, $A^{\dagger}$ is also an *outer inverse* of A [25, pp. 12-14], and the symmetry conditions

$$(AA^{\dagger})^{*} = AA^{\dagger}, \tag{3}$$

and

$$(A^{\dagger}A)^{*} = A^{\dagger}A, \tag{4}$$

where M^{*} denotes the conjugate transpose of the matrix M.

The problem of finding generalized solutions can become delicate in more general spaces, as the set $S(A,y)$ may consist of more than one element or be empty [20,25]; in fact, $G(A,y)$ will be empty if the infimum of the norm of the residual vector is not attained. Of course, there are also many applications of generalized inverses in addition to the solution of linear equations in the singular case [3,21], and this fairly recent subject already has a vast literature [24].

c. The eigenvalue-eigenvector problem.

This problem is posed most naturally in the case $Y = X$ is a Hilbert space with inner product $\langle , \rangle$. One looks for scalars (real or complex numbers) λ and vectors $x \neq 0$ such that

(1.9) $$Ax = \lambda x.$$

Solutions λ of this problem are called *eigenvalues* of the linear operator A; for each eigenvalue λ, nonzero solutions x of (1.9) are said to be the corresponding *eigenvectors* of A. As equation (1.9) is homogeneous in x, the condition $x \neq 0$ may be replaced, for example, by

(1.10) $$\langle x,x \rangle = 1 ,$$

or some other *normalization* condition.

From a standpoint of functional analysis, the determination of the eigenvalues of A is a special case of the more general problem of finding the *spectrum* $\sigma(A)$ of A. In a complex Hilbert space X, the set

(1.11) $$\rho(A) = \{\lambda \mid (A - \lambda I)^{-1} \in L(X,X) \}$$

of complex numbers λ is called the *resolvent* of A. Thus, $\lambda \in \rho(A)$ if and only if the operator $A - \lambda I$ has a continuous inverse. The spectrum of A is simply the complement of the resolvent,

(1.12) $$\sigma(A) = C - \rho(A) ,$$

and hence contains any eigenvalues of A.

d. Linear programming.

In order to formulate this problem, suppose that X,Y are real spaces with *partial ordering* relationships denoted by $\leq$. For most applications, X and Y are taken to be finite-dimensional, in which case the partial ordering is the usual componentwise comparison of vectors [26, pp. 155-158]. Also needed is the *dual space* $X^* = L(X,R)$ of X; that is, the space of continuous *linear functionals* defined on X. It is convenient to use the *bracket notation* of Dirac [7, pp. 18-28] for linear functionals. If $c \in X^*$, then define

(1.13) $$\langle c,x \rangle := c(x) ,$$

which will be consistent with the notation for the inner product if X is a Hilbert space [7, pp. 6-8].

One formulation of the *(primal) linear programming problem* [26, pp. 156-157] is, given $A \in L(X,Y)$, $y \in Y$, $c \in X^*$, find $x \in X$ to *maximize*

(1.14) $$f(x) = \langle c,x \rangle + \xi$$

subject to

(1.15) $$Ax \leq y , \quad x \geq 0 .$$

The function $f(x)$ defined by (1.14) is called the *objective function* of the problem, and conditions (1.15) are known as *constraints*.

Instead of the primal problem (1.14)-(1.15), one may wish to consider the *dual problem* [26, pp. 188-190] which is to find $z \in Y^*$ to *minimize*

$$g(z) = \langle z, y \rangle - \zeta \tag{1.16}$$

subject to

$$A^* z \geq c , \quad z \geq 0 . \tag{1.17}$$

In (1.17), the operator $A^* \in L(Y^*, X^*)$ is the *adjoint* of A, defined by

$$\langle A^* y^*, x \rangle = \langle y^*, Ax \rangle \tag{1.18}$$

for all $y^* \in Y^*$, $x \in X$. It will also be convenient to write

$$y^* A := A^* y^* , \quad y^* \in Y^* ; \tag{1.19}$$

that is, $y^* A$ is the linear functional on X defined by

$$(y^* A)x := y^*(Ax) = \langle y^*, Ax \rangle , \qquad x \in X . \tag{1.20}$$

This is analogous to the notation frequently used in elementary matrix algebra, with x being considered to be a column vector, and y^* a row vector. The scalar quantity (1.20) will also be denoted by

$$\langle y^* Ax \rangle := \langle y^*, Ax \rangle . \tag{1.21}$$

The subject of perturbation methods and theory has a long history, and there is a vast literature devoted to this topic and its applications. The bibliography at the end of this paper, rather than attempting to be comprehensive, lists only references cited in the text, doubtless at the cost of omitting a number of significant contributions.

2. Perturbed linear problems. Perturbation theory, as applied to the linear problems listed in §1, starts from the assumption that their solutions are known for the given *reference data* $A \in L(X,Y)$, $y \in Y$, $c \in X^*$. The object is to study the behavior of these solutions for various classes of *perturbed data*.

$$B = A + \Delta A , \quad z = y + \Delta y , \quad d = c + \Delta c , \tag{2.1}$$

where the *perturbations* $\Delta A \in L(X,Y)$, $\Delta y \in Y$, $\Delta c \in X^*$ or appropriate information about them are given. One then desires to calculate or estimate the corresponding changes $\Delta x \in X$, $\Delta A^{-1} \in L(Y,X)$, $\Delta A^\dagger \in L(Y,X)$, $\Delta\lambda \in \Lambda$ in the solutions $x \in X$, $A^{-1} \in L(Y,X)$, $A^\dagger \in L(Y,X)$, $\lambda \in \Lambda$ of the original problems. Here

$$\Delta A^{-1} = B^{-1} - A^{-1} \tag{2.2}$$

denotes the difference between the inverse, if it exists, of the perturbed operator B and the inverse of the unperturbed operator A, and *not* $(\Delta A)^{-1}$, which may also exist. A similar observation applies to the notation $\Delta A^\dagger$.

As an example, the perturbed linear system

(2.3) $$Bw = z$$

can be solved for

(2.4) $$w = x + \Delta x$$

if Δx can be obtained in terms of ΔA and Δy, the solution x of the unperturbed system (1.1) with reference data A, y being assumed to be known.

The motivation behind perturbation methods is that if the perturbations in the data are "small" in some sense, then one might expect the changes in the solutions to be correspondingly small, at least under suitable conditions. What is referred to here as "small" may vary widely, depending on the specific problem, the type of perturbation considered, the computing power available, and perhaps other factors. In the next section, a framework will be developed to characterize the concept of small perturbations more precisely.

The goals of perturbation theory may be either practical or theoretical. Two uses of perturbation methods in actual computation are to find solutions of perturbed problems with economy of effort, and to obtain error estimates. In the first case, computing the solution of a given linear problem might be extremely laborious, but a large amount of information could be generated in the process. One would then hope to be able to use this information to solve perturbations of the reference problem with less work than required when starting from scratch, as indicated in connection with the illustration (2.3)-(2.4) cited above. In the case of error estimation, the perturbations are considered to arise from inaccuracies in the data and from truncation and roundoff errors in the computation. Usually, these perturbations can only be estimated, and one seeks some kind of information about the possible error in the solution. One approach, called *forward error estimation*, starts from assumptions about the perturbations in the data, and obtains a comparison of the solution actually obtained with that of the reference problem if exact data and computation were employed. For *backward error estimation*, as developed by Wilkinson [34], the solution actually obtained is taken to be the exact solution of some perturbation of the reference problem, and estimates are made of the corresponding changes in the data. With the forward method, the computed solution is considered to be acceptable if it can be shown to be "close" to the (unknown) solution of the reference problem, while in the backward procedure, the criterion of acceptability is that the problem actually solved is "close" to the reference problem in some sense. More precise concepts of "closeness" will be introduced in the next section.

Perturbation methods can also be used for theoretical purposes. If a conceptual framework can be developed in which the problems considered can be viewed as perturbations of problems with known theory, then it may be possible to extend this theory

from one class to the other. This is the basis, for example, of the classical technique of Erhard Schmidt for obtaining the theory of linear Fredholm integral equations of second kind from the theory of finite linear algebraic systems [4, p. 155]. A more general situation will be described in a later section. Another theoretical use of perturbation methods, closely related to error estimation, is to determine the *sensitivity* of the solution of a linear problem to changes in the data. For example, one may wish to know which components of the solution are affected most strongly by a small change in one of the coefficients of the input data, and which are relatively undisturbed. This kind of analysis can also be used to pursue cause-and-effect relationships in mathematical models of various natural systems and processes.

3. Analytic and algebraic perturbations. For the present purposes, it will be convenient to classify perturbations into two nonexclusive categories; analytic and algebraic. This classification arises from the information available in each case and the methodology used to solve the perturbation problem, as well as an attempt to clarify what is meant by a "small" perturbation. In general, analytic perturbation theory uses metric information, and obtains solutions to perturbation problems in terms of series expansions, or by iterative methods. An objective criterion for a perturbation to be small in this case is that the required series or iterations converge. A more subjective condition is that the convergence be rapid enough to be useful in practice. The satisfaction of this restriction will depend, among other things, on the computing power available and whether the transformations involved can be carried out explicitly, or have to be approximated.

The idea of smallness for algebraic perturbations also depends more or less on outside factors. Here, the perturbations of operators are operators with finite-dimensional range, and vectors and functionals are perturbed by elements belonging to finite-dimensional subspaces of the corresponding spaces. The solution of algebraic perturbation problems will require solving finite algebraic problems of similar type, with the judgment as to what constitutes a "small" finite algebraic problem being again tied up with the resources available for computing. For example, early workers in the theory of linear integral equations knew that replacing them by a corresponding finite linear algebraic system would yield good approximate solutions, but despaired of being able to solve systems of order 10 or 20, as might be required to attain the desired accuracy [18, p. 242]. By contrast, today most computing centers are able to furnish the solutions of well-conditioned linear algebraic systems of order 100 or 200 at nominal cost.

More precise formulations will now be made of the type of information given and expected with each type of perturbation.

a. Analytic perturbations.

The fundamental metric information about vectors in Banach spaces $X,Y,\ldots$ is given, of course, by the respective norms $\|\ \|_X$, $\|\ \|_Y,\ldots$. As confusion is unlikely, the subscripts will usually be dropped. If A is a continuous linear operator from X into Y, that is, if $A \in L(X,Y)$, then the numbers

$$M(A) = \sup_{\|x\|=1} \|Ax\|, \quad m(A) = \inf_{\|x\|=1} \|Ax\|, \tag{3.1}$$

exist and are finite [2, p. 54; 16, p. 194] . $M(A)$ and $m(A)$ are called the *upper* and *lower bound* of A, respectively. With the natural definitions of addition and scalar multiplication of linear operators, it is well known [38, p. 163] that $L(X,Y)$ is a Banach space for the *operator norm* $\|A\| = M(A)$. In some spaces, this norm is easy to compute, but in others, finding $M(A)$ might require more effort than solving the problem of interest. For numerical purposes, it is often convenient to assign a norm to the linear operator space $L(X,Y)$ which is easier to compute than the operator norm, and is *consistent* with it in the sense that

$$\|A\| \geq M(A) . \tag{3.2}$$

For example, if $X = Y = E^n$, (complex) n-dimensional Euclidean space, then $A = (a_{ij})$ is represented by an $n \times n$ matrix with eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$. One has

$$M(A) = \max\{|\lambda_1|,|\lambda_2|,\ldots,|\lambda_n|\}, \tag{3.3}$$

which requires finding the eigenvalue of largest modulus of A. On the other hand, the *Euclidean norm* of A,

$$\|A\| = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}} \tag{3.4}$$

is consistent and may be found by a straightforward calculation.

In the case of the adjoint spaces $X^*, Y^*, \ldots$ of continuous linear functionals on $X,Y,\ldots$, the norm will always be defined analogously to the operator norm, that is,

$$\|c\| = \sup_{\|x\|=1} |\langle c,x \rangle| \tag{3.5}$$

for $c \in X^*$.

Thus, in the perturbed linear system (2.3), one would want a convergent process to calculate Δx, or an estimate for $\|\Delta x\|$ in terms of bounds for $\|\Delta A\|$ and $\|\Delta y\|$, and perhaps also the known quantities $\|A\|$, $\|x\|$, $\|y\|$.

b. Algebraic perturbations.

An algebraic perturbation Δy of a vector $y \in Y$ is defined to be an element of a finite-dimensional subspace

(3.6) $$Y_n = span\{y_1, y_2, \ldots, y_n\}$$

of Y consisting of all linear combinations of given independent basis vectors $y_1, y_2, \ldots, y_n$ in Y. A similar definition applies to algebraic perturbations of linear functionals. Ordinarily, algebraic perturbations will be restricted to subspaces with small dimension (in the sense described above). However, if the original spaces are finite-dimensional, it is of course possible to represent an arbitrary perturbation as an algebraic perturbation.

In the case of linear operators, algebraic perturbations are represented by linear operators with finite-dimensional ranges. Such operators are said to be of *finite rank*, or degenerate (in infinite-dimensional spaces). Here, the *dyadic notation* of Dirac [7, pp. 26-28] will be adopted; for $u \in Y$, $v \in X^*$, the symbol $u><v$ will represent an operator of *rank one* from X into Y, with

(3.7) $$(u><v)x = u<v,x> = <v,x>u \in Y$$

for $x \in X$. Also, for $y^* \in Y^*$, the transposed operation will be denoted by

(3.8) $$y^*(u><v) = <y^*,u>\, v \in X^*,$$

again consistent with the notation introduced in §1. In these terms, a general algebraic perturbation $\Delta A \in L(X,Y)$ of *rank* n will be written as

(3.9) $$\Delta A = \sum_{i=1}^{n} u_i><v_i,$$

where the vectors $u_i \in Y$ and functionals $v_i \in X^*$, $i = 1,2,\ldots,n$, form linearly independent sets. The range of the operator (3.9) is $Y_n = span\{u_1, u_2, \ldots, u_n\}$. In the finite-dimensional case, Y_n could coincide with Y, and arbitrary perturbations of linear operators could be written in the form (3.9).

Algebraic perturbations of vectors and linear operators are sometimes referred to as *finite rank modifications*. This terminology is useful if a clear distinction between analytic and algebraic methods is intended. By the use of algebraic perturbation theory, one would expect to obtain the perturbations in solutions of linear problems in the same form as the perturbations in the data. For example, one would want to express Δx as a linear combination of vectors $x_1, x_2, \ldots, x_n$ to be determined, that is, $\Delta x \in span\{x_1, x_2, \ldots, x_n\} = X_n$, a finite-dimensional subspace of X. Similarly, expressions of the form (3.9) for ΔA^{-1} and $\Delta A^{\dagger}$ would be sought. In other words, algebraic perturbations in the data of linear problems are expected to give rise to finite rank modifications of their solutions.

In contrast to analytic perturbation theory, the use of algebraic methods does not involve restrictions on the norms of the perturbations in the data. However, it is possible that algebraic perturbations can be small in the analytic sense, so that either technique could be employed. Also, as illustrated in the next section, certain problems lend themselves to a combination of algebraic and analytic methods.

4. Compact operators and the Fredholm theory. A theoretical application of perturbation methods, which also has implications for numerical computation, is the extension of the theory of finite linear algebraic systems of n equations in n unknowns to certain types of linear equations (1.1) in infinite-dimensional spaces. An extension of this kind will be obtained here by the use of both analytic and algebraic techniques. First, the alternative structure (1.1a) of the theory of equation (1.1) will be given an explicit formulation for the class of operators to be

Definition 4.1. Linear operators belonging to a class $\mathcal{A} \subset L(X,Y)$ are said to have a *Fredholm theory* if for each $A \in \mathcal{A}$, either (i) the *homogeneous equation*

$$Ax = 0 \tag{4.1}$$

has the unique solution $x = 0$, in which case the inhomogeneous equation (1.1) has a unique solution x for each $y \in Y$, or (ii) equation (4.1) has nonzero solutions, each of which can be expressed as a linear combination of a finite number d linearly independent solutions $x_1, x_2, \ldots, x_d \in X$, in which case the *transposed homogeneous equation*

$$zA = 0 \tag{4.2}$$

likewise has d linearly independent solutions $z_1, z_2, \ldots, z_d \in Y^*$, in terms of which all its nonzero solutions are expressible as linear combinations, and the inhomogeneous equation (1.1) has no solutions unless

$$\langle z_i, y \rangle = 0, \quad i = 1,2,\ldots,d. \tag{4.3}$$

If (4.3) is satisfied and x_0 is any solution of (1.1) (sometimes called a *particular solution*), then the *general solution* of the inhomogeneous equation can be written as

$$x = x_0 + \sum_{i=1}^{d} \alpha_i x_i, \tag{4.4}$$

with arbitrary scalars $\alpha_1, \alpha_2, \ldots, \alpha_d$.

For the algebraic case $X = Y = R^n$, real n-dimensional space, the class $\mathcal{A}$ of linear operators with Fredholm theory consists of all $n \times n$ real matrices $A = (a_{ij})$, that is, $\mathcal{A} = L(R^n, R^n)$, and the alternatives in Definition 4.1 were known to hold long before 1903, when the Norwegian mathematician Ivar Fredholm [6] established the correspondence between the theories of finite linear algebraic systems and linear integral equations of the form

$$x(s) - \lambda \int_0^1 K(s,t)\, x(t)\,dt = y(s), \quad 0 \leq s \leq 1, \tag{4.5}$$

giving rise to the present name for the theory.

Definition 4.2. A linear operator $K \in L(X,Y)$ is said to be compact if, given any $\varepsilon > 0$, there exists a positive integer $n = n(\varepsilon)$ such that

$$K = S + F, \tag{4.6}$$

where $\|S\| < \varepsilon$ and F is of finite rank n.

A compact operator may thus be regarded as a small analytic perturbation of an operator of finite rank, or as a finite rank modification of an operator which is small in the analytic sense. It will be shown that the Fredholm theory can be extended to operators which can be expressed as the sum of a linear operator having a continuous inverse and a compact operator. That is, if $\mathcal{J} \subset L(X,Y)$ denotes the class of linear operators J such that $J^{-1} \in L(Y,X)$ exists, $\mathcal{K} \subset L(X,Y)$ the class of compact operators, and $\mathcal{A} = \mathcal{J} \oplus \mathcal{K}$ the class of linear operators of the form

$$A = J + K, \quad J \in \mathcal{J}, \quad K \in \mathcal{K} \tag{4.7}$$

then each $A \in \mathcal{A}$ has a Fredholm theory. This assertion will be proved in the next section by combining results from both analytic and algebraic perturbation theory. First, it will be shown that if $J \in \mathcal{J}$, then one has the well known result that $J + \Delta J \in \mathcal{J}$ for $\|\Delta J\|$ sufficiently small. Later, the *Fredholm alternative* given in Definition 4.1 will be established for operators which are the sum of invertible linear operators and linear operators of finite rank. The statement that operators of the form (4.7) have a Fredholm theory will then follow from Definition 4.2.

5. Nonsingular linear equations and operators. In this section, the problems of solving linear systems and the inversion of linear operators will be considered for the nonsingular case. Here, alternative (1.1a(i)) holds, and the inverse A^{-1} of the operator A exists.

a. Analytic perturbation of well-posed problems.

Definition 5.1. A problem is said to be *well-posed* if it has a unique solution which depends continuously on the data.

As a general rule, analytic perturbation methods are only successful when applied to well-posed problems. This can require the imposition of additional conditions on the data to insure uniqueness and continuous dependence of the solution, at least in some neighborhood of the solution of the reference problem. For the linear problems considered in this section to be well-posed, the continuity (and hence boundedness) of A^{-1} is required in addition to its existence. Consequently, it will be assumed that $A^{-1} \in L(Y,X)$ in the following discussion of the application of analytic perturbation theory. If A maps X onto Y, then it is well known that $A^{-1} \in L(Y,X)$ if and only if $m(A) > 0$ [2, pp. 145-150]. Lonseth [16, p. 194] has derived the relationship

$$m(A)M(A^{-1}) = M(A)m(A^{-1}) = 1 \tag{5.1}$$

between the upper and lower bounds of a linear operator A with the continuous inverse A^{-1}. Furthermore, $(A + \Delta A)^{-1}$ exists if $M(\Delta A) < m(A)$. Using (5.1), this result may be stated in terms of consistent norms.

Theorem 5.1. If $\| \Delta A \| < \dfrac{1}{\| A^{-1} \|}$, then $(A+\Delta A)^{-1}$ exists and is given by

$$(5.2) \qquad (A+\Delta A)^{-1} = \sum_{n=0}^{\infty} (-A^{-1}\Delta A)^n A^{-1} .$$

Proof: The hypothesis guarantees the convergence of the *Neumann series* on the right side of (5.2). Denoting this series by S, one finds by direct manipulation that $(A+\Delta A)S = I_Y$, the identity operator in Y, and $S(A+\Delta A) = I_X$; hence, $S = (A+\Delta A)^{-1}$. QED

Although the Neumann series expansion (5.2) is useful for theoretical purposes, it is likely to be too slowly convergent for practical computation. The partial sums

$$(5.3) \qquad S_k = \sum_{n=0}^{k} (-A^{-1}\Delta A)^n A^{-1}$$

of the Neumann series (5.2) may be obtained by the simple iteration

$$(5.4) \qquad S_0 = A^{-1}, \quad S_k = S_0 - (A^{-1}\Delta A)A_{k-1}, \quad k = 1,2,\ldots .$$

From (5.2), for $\theta = \| A^{-1}\Delta A \|$,

$$(5.5) \qquad \| (A+\Delta A)^{-1} - S_k \| \leq \frac{\theta^{k+1}}{1-\theta} \| A^{-1} \|.$$

In order to find a more efficient method, the *Hotelling-Lonseth algorithm* [17] may be adapted to this purpose. In this special case, the iteration process is

$$(5.6) \qquad B_0 = A^{-1}, \quad B_k = [1 + (-A^{-1}\Delta A)^{2^{k+1}}]B_{k-1}, \quad k = 1,2,\ldots .$$

It is easy to show by mathematical induction that $B_k = S_{2^k-1}$; hence, from (5.5),

$$(5.7) \qquad \|(A+\Delta A)^{-1} - B_k\| \leq \frac{\theta^{2^k}}{1-\theta} \| A^{-1} \| ,$$

so that the sequence $\{B_k\}$ defined by (5.6) converges quadratically to $(A+\Delta A)^{-1}$. The only additional labor required over the more slowly convergent algorithm (5.4) is the repeated squaring of the small operator $-A^{-1}\Delta A$.

Attention will now be devoted to the estimation of the perturbations ΔA^{-1} and Δx in the inverse of the perturbed operator and the solution of the perturbed linear equation (2.3), respectively [14, 15, 16]. It will be helpful to introduce the notion of the *condition number* of a bounded linear operator. For $A \in L(X,Y)$, the *exact* condition number $K(A)$ of A is defined to be

$$(5.8) \qquad K(A) = \frac{M(A)}{m(A)} ,$$

and is a measure of the distortion of the image in Y of the unit ball in X as transformed by the operator A. If A has a continuous inverse, then $K(A) = M(A)M(A^{-1})$ by (5.1). For computational purposes, it may be expedient to use

consistent norms for $L(X,Y)$ and $L(Y,X)$, and the *approximate* condition number

(5.9) $$k(A) = \|A\| \cdot \|A^{-1}\| ,$$

which is an upper bound for $\kappa(A)$. The inequalities given below will be stated in terms of consistent norms and approximate condition numbers, but remain valid if these upper bounds are replaced by their exact values.

First, from (5.2),

(5.10) $$\Delta A^{-1} = (A+\Delta A)^{-1} - A^{-1} = \sum_{n=1}^{\infty} (-A^{-1}\Delta A)^n A^{-1} ,$$

and thus,

(5.11) $$\|\Delta A^{-1}\| \leq \frac{\|A^{-1}\|^2 \, \|\Delta A\|}{1 - \|A^{-1}\| \cdot \|\Delta A\|} .$$

Dividing (5.11) by $\|A^{-1}\|$ and multiplying and dividing $\|\Delta A\|$ on the right hand side by $\|A\|$ gives

(5.12) $$\frac{\|\Delta A^{-1}\|}{\|A^{-1}\|} \leq \frac{k(A)\dfrac{\|\Delta A\|}{\|A\|}}{1-k(A)\dfrac{\|\Delta A\|}{\|A\|}} ,$$

which expresses the relative change in the inverse in terms of the relative perturbation of the reference operator and its (approximate) condition number. A similar expression will now be obtained for the perturbation Δx in the solution of (2.3).

Theorem 5.2. If $\|\Delta A\| < \dfrac{1}{\|A^{-1}\|}$, then the perturbed linear equation (2.3) has a unique solution $w = x + \Delta x$ for each $z = y + \Delta y$, and

(5.13) $$\frac{\|\Delta x\|}{\|x\|} \leq \frac{k(A)}{1 - k(A)\dfrac{\|\Delta A\|}{\|A\|}} \left[\frac{\|\Delta A\|}{\|A\|} + \frac{\|\Delta y\|}{\|y\|} \right]$$

provided, of course, that $y \neq 0$.

Proof: By Theorem 5.1, the hypothesis guarantees that $B^{-1} = (A+\Delta A)^{-1} = A^{-1}+\Delta A^{-1}$ exists, which implies the unique solvability of (2.3) for each z. Writing (2.3) as

(5.14) $$(A + \Delta A)(x + \Delta x) = y + \Delta y ,$$

one obtains

(5.15) $$\Delta x = \Delta A^{-1} y + (A+\Delta A)^{-1}\Delta y .$$

As $y = Ax$, from (5.10),

(5.16) $$\Delta A^{-1} y = \sum_{n=1}^{\infty} (-A^{-1}\Delta A)^n x ,$$

so that

$$(5.17)\qquad \|\Delta A^{-1}y\| \leq \frac{\|A^{-1}\|\cdot\|\Delta A\|\cdot\|x\|}{1-\|A^{-1}\|\cdot\|\Delta A\|} = \frac{k(A)}{1-k(A)\frac{\|\Delta A\|}{\|A\|}}\cdot\frac{\|\Delta A\|}{\|A\|}\cdot\|x\|$$

Similarly, from (5.2) and the fact that $\|y\| = \|Ax\| \leq \|A\|\cdot\|x\|$,

$$(5.18)\qquad \|(A+\Delta A)^{-1}\Delta y\| \leq \frac{\|A^{-1}\|\cdot\|\Delta y\|}{1-\|A^{-1}\|\cdot\|\Delta A\|}\cdot\frac{\|A\|\cdot\|x\|}{\|y\|} =$$

$$\frac{k(A)}{1-k(A)\frac{\|\Delta A\|}{\|A\|}}\cdot\frac{\|\Delta y\|}{\|y\|}\cdot\|x\| .$$

Inequality (5.13) now follows directly from (5.15), (5.17), and (5.18). QED

b. Algebraic perturbation of nonsingular linear equations and operators.

The simplest type of algebraic perturbation (2.3) of the linear system (1.1) is with $\Delta A = 0$ and Δy restricted to belong to a finite-dimensional subspace Y_n of Y. Given a basis $\{y_1, y_2, \ldots, y_n\}$ for Y_n, one need only find the corresponding basis vectors

$$(5.19)\qquad x_i = A^{-1}y_i , \qquad i = 1,2,\ldots,n,$$

of the subspace $X_n \subset X$ which will then contain all possible perturbations Δx. Thus, given

$$(5.20)\qquad \Delta y = \alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_n y_n ,$$

it follows that

$$(5.21)\qquad \Delta x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n .$$

In actual computation, it may be more efficient to solve the n systems $Ax_i = y_i$, $i = 1,2,\ldots,n$, for the basis vectors for X_n, even if X is finite-dimensional [5, p. 77], than to calculate A^{-1}.

To introduce the study of the effect of a finite-rank modification of an operator upon its inverse, the case of rank one perturbation will be considered first, as all the indicated operations can be displayed explicitly. For $\Delta A = u><v$ with $u \in Y$, $v \in X^*$ nonzero, the solvability of the perturbed system (2.3), that is

$$(5.22)\qquad (A + u><v)w = z ,$$

will be investigated for arbitrary z. As A^{-1} is assumed to exist, the equations $A\hat{u} = u$, $A\hat{z} = z$ can be solved uniquely for $\hat{u} = A^{-1}u$, $\hat{z} = A^{-1}z$, respectively. In terms of these solutions, (5.22) may be written as

$$(5.23)\qquad w = \hat{z} - \hat{u}<v,w> .$$

The key to the solvability of (5.23), and hence of (5.22), is the determination of the number $\xi = \langle v,w \rangle$. From (5.23),

$$\langle v,w \rangle + \langle v,\hat{u} \rangle \langle v,w \rangle = \langle v,\hat{z} \rangle \tag{5.24}$$

If the *determinant*

$$\delta = 1 + \langle v,\hat{u} \rangle = 1 + \langle v A^{-1} u \rangle \tag{5.25}$$

does not vanish, then (5.24) has the unique solution

$$\langle v,w \rangle = \frac{\langle v,\hat{z} \rangle}{\delta} = \frac{\langle v A^{-1} z \rangle}{1+\langle v A^{-1} u \rangle}, \tag{5.26}$$

where the notation (1.21) has been used in (5.25) and (5.26). Substitution of (5.26) into (5.23) yields

$$w = A^{-1} z - A^{-1} u \frac{\langle v A^{-1} z \rangle}{\delta} = \left(A^{-1} - \frac{A^{-1} u \rangle \langle v A^{-1}}{1+\langle v A^{-1} u \rangle} \right) z, \tag{5.27}$$

so that

$$(A + u\rangle\langle v)^{-1} = A^{-1} - \frac{A^{-1} u \rangle \langle v A^{-1}}{1 + \langle v A^{-1} u \rangle}, \tag{5.28}$$

provided $\delta \neq 0$. Hence, the inverse of a rank one modification of an invertible operator, if it exists, is a rank one modification of the inverse of the reference operator. The symmetry of (5.28), sometimes called the *Sherman-Morrison-Woodbury formula* [11, pp. 123-124; 35, 46], is appealing.

Using (5.28), the solution $w = x + \Delta x$ *of* (5.22) is

$$x + \Delta x = x + A^{-1}\Delta y - \frac{\langle v, x + A^{-1}\Delta y \rangle}{1+\langle v A^{-1} u \rangle} A^{-1} u, \tag{5.29}$$

or

$$\Delta x = A^{-1}\Delta y - \frac{\langle v, x + A^{-1} y \rangle}{1+\langle v A^{-1} u \rangle} A^{-1} u. \tag{5.30}$$

Thus, the perturbation Δx is a linear combination of $A^{-1}\Delta y$ and the vector $\hat{u} = A^{-1}u$. If Δy is an algebraic perturbation of the form (5.20), and $\hat{u}$ is independent of the vectors $x_i = A^{-1} y_i$, $i = 1,2,\ldots,n$, then Δx will lie in the $(n+1)$-dimensional subspace $X_{n+1} = span\{\hat{u}, x_1, x_2, \ldots, x_n\}$ of X; otherwise $\Delta x \in X_n = span\{x_1, x_2, \ldots, x_n\}$.

Before going to the general case, two applications of algebraic perturbation theory will be given which involve rank one modifications. The first is to the Fredholm integral equation (4.5) in which the *kernel* $K(s,t)$ has the special form

$$K(s,t) = \begin{cases} u(t)v(s), & 0 \leq t \leq s \leq 1, \\ u(s)v(t), & 0 \leq s \leq t \leq 1. \end{cases} \tag{5.31}$$

This type of kernel arises in applications; for example, as a Green's function determined by a two-point boundary value problem [32]. Given the representation (5.31) for $K(s,t)$, the integral equation (4.5) may be written as

$$x(s) - \lambda \int_0^s L(s,t)\, x(t)dt - \lambda \int_0^1 u(s)v(t)x(t)dt = y(s) , \tag{5.32}$$

where

$$L(s,t) = u(t)v(s) - u(s)v(t), \qquad 0 \leq t \leq s \leq 1 . \tag{5.33}$$

Equation (5.32) is of the form $(A - \lambda u >< v)x = y$, where $A = I - \lambda L$ is a linear Volterra integral operator of second kind with kernel (5.33), and $u >< v$ is a Fredholm integral operator of first kind and rank one with kernel $u(s)v(t)$. The inverse $A^{-1} = (I - \lambda L)^{-1}$ of the Volterra operator of second kind exists for all λ [30, pp. 52-53], and thus the linear Volterra integral equation

$$\hat{w}(s) - \lambda \int_0^s L(s,t)\, \hat{w}(t)dt = w(s) \tag{5.34}$$

can be solved for arbitrary $w(s)$; in particular, one obtains $\hat{w}(s) = \hat{u}(s)$ for $w(s) = u(s)$, and $\hat{w}(s) = \hat{y}(s)$ for $w(s) = y(s)$. Corresponding to (5.25), if the *Fredholm determinant*

$$\delta = 1 - \lambda <v,\hat{u}> = 1 - \lambda \int_0^1 v(t)\hat{u}(t)dt \tag{5.35}$$

does not vanish, then, from (5.27),

$$x(s) = \hat{y}(s) + \frac{\lambda}{\delta}\, \hat{u}(s) \int_0^1 v(t)\hat{y}(t)dt \tag{5.36}$$

is the unique solution of (5.32). Hence, the solution of the Fredholm integral equation (4.5) with the kernel (5.31) *can be obtained* by solving the Volterra integral equation (5.34) with right-hand sides $w(s) = u(s)$ and $w(s) = y(s)$, followed by the calculation of the inner product integrals in (5.35) and (5.36).

The second application to be considered for rank one modification of a linear operator is to backward error analysis in the solution of linear equations. Suppose that one attempts to solve the linear equation (1.1) and obtains, instead of x , an approximate solution w such that

$$A w = y + r , \tag{5.37}$$

with *nonzero residual* r , The Hahn-Banach theorem [38, p. 186] guarantees the existence of a linear functional $w^* \in X^*$ such that $\| w^* \| = 1$ and $<w^*,w> = \| w \|$. Thus, w is the *exact* solution of the linear equation

$$\left(A - \frac{r >< w^*}{\| w \|} \right) w = y \tag{5.38}$$

with perturbed operator and desired right-hand side. An analytic bound for the perturbation of A is thus

$$\| \Delta A \| = \frac{\| r >< w^* \|}{\| w \|} = \frac{\| r \|}{\| w \|} . \tag{5.39}$$

Returning to the study of general algebraic perturbations, note that the equivalence of (5.22) and the single scalar equation (5.24) establishes that (5.22) has a Fredholm theory, because (5.24) does. In the case $\delta = 0$, the homogeneous equation $(A + u >< v)w = 0$ is satisfied by (and only by) vectors $w = \alpha\hat{u}$ with α arbitrary. The inhomogeneous equations (5.22) and (5.24) then have solutions only if

$$< v,\hat{z}> = <vA^{-1}z > = <\hat{v},z> = 0 , \tag{5.40}$$

where $\hat{v} = vA^{-1}$ satisfies the transposed homogeneous equation

$$\hat{v}(A + u >< v) = 0 . \tag{5.41}$$

Theorem 5.3. If $\mathcal{I} \subset L(X,Y)$ denotes the class of all invertible linear operators, and $\mathcal{F} \subset L(X,Y)$ the class of all linear operators of finite rank, then all linear operators belonging to the class $\mathcal{A} = \mathcal{I} \oplus \mathcal{F}$ have a Fredholm theory.

Proof: If $B \in \mathcal{A} = \mathcal{I} \oplus \mathcal{F}$, then there is an ivertible linear operator $A \in \mathcal{I}$ for which B can be written as

$$B = A + \sum_{j=1}^{n} u_j >< v_j , \tag{5.42}$$

where $u_j \in Y$, $v_j \in X^*$, $j = 1,2,\ldots,n$, are linearly independent sets of vectors and functionals, respectively. Equation (2.3) in this case is equivalent to

$$w = \hat{z} - \sum_{j=1}^{n} \hat{u}_j <v_j,w> , \tag{5.43}$$

where $\hat{z} = A^{-1}z$, $\hat{u}_j = A^{-1}u_j$, $j = 1,2,\ldots,n$. Applying the functionals $v_1, v_2, \ldots,$ to (5.43) in turn gives the equivalent finite linear algebraic system of equations

$$\xi_i + \sum_{j=1}^{n} \alpha_{ij}\xi_j = \zeta_i , \quad i = 1,2,\ldots,n \tag{5.44}$$

for $\xi_i = <v_i,w>$, where $\zeta_i = < v_i,\hat{z}>$ and $\alpha_{ij} = < v_i,\hat{u}_j>$, $i,j = 1,2,\ldots,n$. As the Fredholm alternative applies to (5.44), it follows that operators of the form (5.42) have a Fredholm theory. QED

In the nonsingular case, an expression can be obtained for B^{-1} as a finite rank modification of A^{-1}. Let

$$M = (\delta_{ij} + \alpha_{ij}) \tag{5.45}$$

debite the matrix of coefficients of the linear system (5.44), where δ_{ij} is the Kronecker delta: $\delta_{ij} = 0$ if $i \neq j$, $\delta_{ii} = 1$. As the determinant δ of M is assumed to be nonzero, the inverse of M may be written

$$M^{-1} = \frac{1}{\delta} (\beta_{ij}) , \tag{5.46}$$

and thus

$$\xi_i = < v_i,w> = \frac{1}{\delta} \sum_{j=1}^{n} \beta_{ij}\zeta_j = \frac{1}{\delta} \sum_{j=1}^{n} \beta_{ij} < v_j,\hat{z}> , \tag{5.47}$$

$i = 1,2,\ldots,n$. Using (5.43) and the fact that $< v_j,\hat{z}> = < v_jA^{-1}z> = < \hat{v}_j,z >$ for

$\hat{v}_j = v_j A^{-1}$, $i = 1,2,\ldots,n$, one obtains the solution w of (2.3) in this case as

$$w = \left(A^{-1} - \frac{1}{\delta} \sum_{i=1}^{n} \sum_{j=1}^{n} A^{-1} u_i > \beta_{ij} < v_j A^{-1} \right) z \,. \tag{5.48}$$

By taking appropriate linear combinations $\tilde{u}_1,\tilde{u}_2,\ldots,\tilde{u}_n$ of $u_1,u_2,\ldots,u_n$ and $\tilde{v}_1,\tilde{v}_2,\ldots,\tilde{v}_n$ of $v_1,v_2,\ldots,v_n$ (for example, by an *LU-decomposition* of M^{-1} [5, pp. 27-32]), (5.48) may be put in the form

$$w = \left(A^{-1} - \frac{1}{\delta} \sum_{j=1}^{n} A^{-1} \tilde{u}_j >< \tilde{v}_j A^{-1} \right) z \,, \tag{5.49}$$

from which

$$\left(A + \sum_{j=1}^{n} u_j >< v_j \right)^{-1} = A^{-1} - \frac{1}{\delta} \sum_{j=1}^{n} A^{-1} \tilde{u}_j >< \tilde{v}_j A^{-1} \,, \tag{5.50}$$

which is analogous to (5.28).

Another way to find the inverse of the perturbed operator (5.42) is the method of *successive rank one modifications*, which does not require obtaining M^{-1} explicitly. Set

$$B_0 = A \,, \qquad B_0^{-1} = A^{-1} \,, \tag{5.51}$$

and then the algorithm

$$\begin{cases} B_k = B_{k-1} + u_k >< v_k \,, \\ B_k^{-1} = B_{k-1}^{-1} - \dfrac{B_{k-1}^{-1} u_k >< v_k B_{k-1}^{-1}}{1 + < v_k B_{k-1}^{-1} u_k >} \,, \end{cases} \tag{5.52}$$

$k = 1,2,\ldots,n$ will give $B^{-1} = B_n^{-1}$ if none of the *intermediate determinants*

$$\delta_k = 1 + < v_k B_{k-1}^{-1} u_k > \,, \quad k = 1,2,\ldots,n \,, \tag{5.53}$$

vanish.

It should be noted again that it is not necessary to obtain A^{-1} to solve the equation (2.3) for

$$w = \hat{z} - \sum_{j=1}^{n} \xi_j \hat{u}_j \,, \tag{5.54}$$

as given by (5.43). What is required is to solve equation (1.1) for the $n+1$ right-hand sides $y = z, u_1,u_2,\ldots,u_n$ for $x = \hat{z},\hat{u}_1,\hat{u}_2,\ldots,\hat{u}_n$, calculate the coefficients of the system (5.44) of n equations for the n unknowns $\xi_1,\xi_2,\ldots,\xi_n$, solve this system, and then form the linear combination (5.54).

An important application of the above technique of algebraic perturbation is to the numerical solution of partial differential equations by what is called the *capacitance matrix method* [36, 42]. The basic problem is to solve, for example, the Poisson or Helmholtz equation on a region Ω, with information given on its boundary $\partial\Omega$ (see Figure 5.1). The use of finite-difference methods will lead to a linear

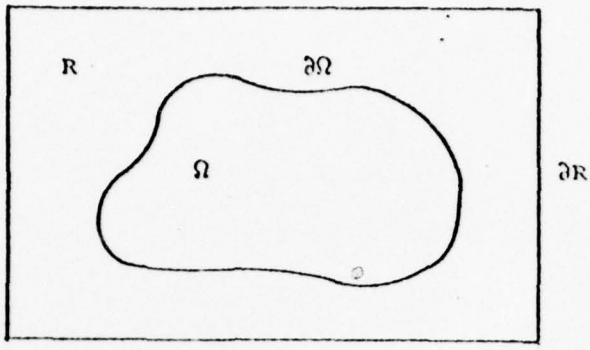

Figure 5.1

algebraic system $B w = z$ which may be very laborious to solve. On the other hand, rapid and effective methods may be available for the algebraic system $Ax = y$ corresponding to the same finite-difference approximation to the problem posed on an enclosing rectangle R with boundary ∂R. By regarding the algebraic system obtained for Ω as a finite rank perturbation of the easily solved system arising from the approximate problem on R, a considerable reduction in effort may be possible. Typically, if the order of the systems (1.1) and (2.3) is about n^2, then the rank of the perturbations ΔA and Δy will be approximately n.

The Fredholm theory will now be shown to apply to operators which are the sum of a continuously invertible operator and a compact operator.

Theorem 5.4. Operators A belonging to the class $\mathcal{A}$ defined by (4.7) have a Fredholm theory.

Proof: Choose $\epsilon < 1/\| J^{-1} \|$. According to Definition 4.2, the compact operatio K may be written as

$$K = S + \sum_{j=1}^{n} u_j > < v_j \,, \tag{5.55}$$

where $n = n(\epsilon)$ is finite. Thus,

$$A = J + S + \sum_{j=1}^{n} u_j >< v_j \,, \tag{5.56}$$

and Theorem 5.1 guarantees the existence of the inverse operator $(J+S)^{-1} \in L(Y,X)$. It follows from Theorem 5.3 that A has a Fredholm theory. QED

Theorem 5.4 provides a basis for the "kernel splitting" method due to Erhard Schmidt [4, p. 155] for proving the Fredholm Alternative Theorem [6] for the linear integral equation (4.5). Suppose that $K(s,t)$ is continuous, or at least can be approximated sufficiently well by a kernel of finite rank so that one can write

(5.57) $$K(s,t) = S(s,t) + \sum_{j=1}^{n} u_j(s)v_j(t) ,$$

where $S(s,t)$ is the kernel of a linear integral operator S with $\| S \| < \frac{1}{|\lambda|}$ in the appropriate norm. Then, the linear integral operator $I - \lambda K$ in (4.5) has the form

(5.58) $$I - \lambda K = I - \lambda S - \lambda \sum_{j=1}^{n} u_j >< v_j ,$$

where $(I - \lambda S)^{-1} = T(\lambda)$ exists by Theorem 5.1. Applying this operator to the equation $(I - \lambda K)x = y$, the integral equation (4.5) is seen to be equivalent to the linear equation

(5.59) $$(I - \lambda \sum_{j=1}^{n} T(\lambda)u_j >< v_j)x = T(\lambda)y,$$

and thus $I - \lambda K$ has a Fredholm theory by Theorem 5.3. This approach regards $I-\lambda K$ as an algebraic perturbation of the invertible operator $I-\lambda S$.

On the other hand, suppose that

(5.60) $$(I - \lambda \sum_{j-1}^{n} u_j >< v_j)^{-1} = I + \frac{\lambda}{\delta} \sum_{j=1}^{n} \tilde{u}_j >< \tilde{v}_j = Z(\lambda)$$

exists, where the notation (5.50) has been used. Then, $I - \lambda K$ is an analytic perturbation of an invertible operator, and (4.5) is equivalent to the equation

(5.61) $$(I - \lambda Z(\lambda)S)x = Z(\lambda)y .$$

From Theorem 5.1, if $(I - \lambda K)^{-1}$ exists and $\| \lambda S\| < 1/\|(I- \lambda K)^{-1}\|$, then $Z(\lambda)$ exists, so all sufficiently good finite rank approximations (5.57) to $K(s,t)$ will lead to a solvable perturbed equation

(5.62) $$(I - \lambda \sum_{j=1}^{n} u_j >< v_j)w = y$$

which is equivalent to a finite linear algebraic system of the form (5.44). Conversly, if the inverse operator $Z(\lambda)$ exists and $\| \lambda Z(\lambda)S \| < 1$, then it follows from the same theorem that $(I - \lambda K)^{-1}$ exists. If $u_1,u_2,\ldots,u_n$ and $v_1,v_2,\ldots,v_n$ are chosen so that all the inner products required can be calculated explicitly, then this gives a method for concluding the existence and uniqueness of the solution of the integral equation (4.5) on the basis of a finite set of algebraic computations, as well as a technique to obtain approximate solutions. An error analysis for (5.62) may be carried out by the analytic methods of §5a with $\Delta\lambda = -\lambda S$, $\Delta y = 0$. A similar approach can be used on (5.59) with $T(\lambda)$ replaced by

(5.63) $$T_k(\lambda) = I + \lambda S + \lambda^2 S^2 + \ldots + \lambda^k S^k .$$

Setting $z = T_k(\lambda)y$, the perturbed equation

(5.64) $$(I - \lambda \sum_{j=1}^{n} T_k(\lambda)u_j >< v_j)w = z$$

may be analyzed by the same technique, with

$$\Delta A = \lambda \sum_{j=1}^{n} (\lambda S)^{k+1} T(\lambda) u_j >< v_j , \tag{5.65}$$

and

$$\Delta T(\lambda) y = -(\lambda S)^{k+1} T(\lambda) y . \tag{5.66}$$

As $\| T(\lambda) \| \leq 1/(1- \| \lambda S \|)$, and for

$$\lambda F = \lambda \sum_{j=1}^{n} u_j >< v_j , \tag{5.67}$$

one has

$$\| \Delta A \| \leq \frac{\| \lambda S \|^{k+1}}{1- \| \lambda S \|} \| \lambda F \| , \quad \| \Delta T(\lambda) y \| \leq \frac{\| \lambda S \|^{k+1}}{1- \| \lambda S \|} \| y \| \tag{5.68}$$

with

$$\| \lambda F \| \leq |\lambda| \sum_{j=1}^{n} \| u_j \| \cdot \| v_j \| \tag{5.69}$$

from (5.67).

Another analytic approach to the approximate solution of (4.5) which also yields error bounds is to solve (5.61) by iteration, as $\| \lambda z(\lambda) S \| < 1$ [29].

6. The singular case and generalized inverses. Attention will now be devoted to linear problems which are ill-posed because the linear operator involved does not have a bounded inverse. As the solutions, if any, of ill-posed problems do not depend on the data in a continuous fashion, it might be expected in this situation that analytic perturbation methods will be of little utility, or can be applied only under very restrictive conditions. For example, there is an inherent limitation as to how well an operator $B \in L(X,Y)$ without a continuous inverse can be approximated by an operator A belonging to the class $\mathcal{G} \subset L(X,Y)$ of operators with continuous inverses $A^{-1} \subset L(Y,X)$. From Theorem 5.1,

$$\| B - A \| = \| \Delta A \| \geq \frac{1}{\| A^{-1} \|} , \tag{6.1}$$

otherwise, the assumption that $B \notin \mathcal{G}$ would be contradicted. Also, from (6.1),

$$\| A^{-1} \| \geq \frac{1}{\| B-A \|} = \frac{1}{\| \Delta A \|} , \tag{6.2}$$

so that $\| A^{-1} \|$ and the approximate condition number

$$k(A) \geq \frac{\| A \|}{\| \Delta A \|} \geq \left| \frac{\| B \|}{\| \Delta A \|} - 1 \right| \tag{6.3}$$

grow without limit as $\| \Delta A \| \to 0$. Clearly, computational difficulties can be expecpected in the calculation of A^{-1} or in the solution of the linear equation (1.1) if A is very close in the analytic sense to an operator B which does not have a continuous inverse.

Theorem 6.1. If $\{A_n\} \subset \mathcal{J}$ is any sequence of linear operators such that $\lim_{n \to \infty} \|A_n - B\| = 0$, then $B \notin \mathcal{J}$ if and only if (6.2) holds for each $A = A_n$, $n = 1,2,\ldots$.

Proof: If $B \notin \mathcal{J}$, then it has already been shown that (6.2) holds for each A_n. To show the converse, suppose that $B \in \mathcal{J}$, and choose n sufficiently large so that $\|\Delta A_n\| = \|A_n - B\| < 1/2\|B^{-1}\|$. It then follows from (5.2) that

$$\|A_n^{-1}\| \leq \frac{\|B^{-1}\|}{1 - \|\Delta A_n\| \cdot \|B^{-1}\|} < \frac{1}{\|\Delta A_n\|}, \tag{6.4}$$

a contradiction of (6.2) which proves the theorem. QED

An evident drawback of analytic perturbation theory is that, in general, no conclusions can be drawn from the existence of $A^{-1} \in L(Y,X)$ about the invertibility or noninvertibility of any operator B for which inequality (6.1) holds. The algebraic theory, on the other hand, states that if B is the finite rank modification (5.42) of an invertible linear operator $A \in \mathcal{A}$, then B^{-1} exists if and only if

$$\delta = \det(\delta_{ij} + \langle v_i A^{-1} u_j \rangle) \neq 0. \tag{6.5}$$

Of course, one would still expect computational difficulty if B is nearly singular, especially if the inner products $\alpha_{ij} = \langle v_i A^{-1} u_j \rangle$, $i,j = 1,2,\ldots,n$, can only be calculated approximately.

The algebraic approach also provides information in the singular case. Supposing that $\delta = 0$, consider the transposed homogeneous equation

$$t(A + \sum_{i=1}^{n} u_i \rangle\langle v_i) = 0 \tag{6.6}$$

for $t \in X^*$. Using the technique of §5b, this is equivalent to the finite linear algebraic system

$$\tau_j + \sum_{i=1}^{n} \tau_i \alpha_{ij} = 0, \quad j = 1,2,\ldots,n, \tag{6.7}$$

for $\tau_j = \langle t, u_j \rangle$. The system of equations (6.7) is the transposed homogeneous system corresponding to (5.44). If $\delta = 0$, then (6.7) has d linearly independent solutions

$$\tau^{(k)} = (\tau_1^{(k)}, \tau_2^{(k)}, \ldots, \tau_n^{(k)}), \quad k = 1,2,\ldots,d, \tag{6.8}$$

and, corresponding to these, equation (6.6) also has d linearly independent solutions

$$t^{(k)} = \sum_{i=1}^{n} \tau_i^{(k)} v_i A^{-1} = \sum_{i=1}^{n} \tau_i^{(k)} \hat{v}_i, \tag{6.9}$$

$k = 1,2,\ldots,d$. Likewise, the homogeneous system

$$\xi_i + \sum_{j=1}^{n} \alpha_{ij} \xi_j = 0, \quad i = 1,2,\ldots,n, \tag{6.10}$$

has d linearly independent solutions

$$\zeta^{(k)} = (\zeta_1^{(k)},\zeta_2^{(k)},\ldots,\zeta_n^{(k)})^T, \quad k = 1,2,\ldots,d, \tag{6.11}$$

from which are obtained the corresponding linearly independent solutions

$$w^{(k)} = \sum_{j=1}^{n} \zeta_j^{(k)} A^{-1} u_j = \sum_{j=1}^{n} \zeta_j^{(k)} \hat{u}_j, \tag{6.12}$$

$k = 1,2,\ldots,d$, of the homogeneous equation

$$(A + \sum_{j=1}^{n} u_j >< v_j)w = 0. \tag{6.13}$$

Representing the right-hand sides of the system (5.44) as the vector

$$\zeta = (\zeta_1,\zeta_2,\ldots,\zeta_n)^T = (<v_1 A^{-1} z>,<v_2 A^{-1} z>,\ldots,<v_n A^{-1} z>)^T, \tag{6.14}$$

it is seen immediately that the conditions for the solvability of the finite inhomogeneous system (5.44) and the equivalent inhomogeneous equation (2.3) for the case $\delta = 0$ are

$$<\tau^{(k)},\zeta> = < \sum_{i=1}^{n} <\tau_i^{(k)} \hat{v}_i, z> = < t^{(k)}, z> = 0, \tag{6.15}$$

$k = 1,2,\ldots,d$; that is, z must be orthogonal to all solutions of the homogeneous equation (6.6). If (6.15) is satisfied, then the *general solution* of (2.3) may be written as

$$w = \hat{w} + \sum_{k=1}^{d} \alpha_k w^{(k)}, \tag{6.16}$$

where $\hat{w}$ is some *particular solution* of (2.3), and the *complementary vectors*

$$\tilde{w} = \tilde{w}(\alpha_1,\alpha_2,\ldots,\alpha_k) = \sum_{k=1}^{d} \alpha_k w^{(k)} \tag{6.17}$$

satisfy the homogeneous equation (6.13) for arbitrary $\alpha_1,\alpha_2,\ldots,\alpha_k$.

Usually, in actual computational solution of linear equations, the distinction between the singular and nonsingular cases is not as clear-cut as in the alternative (1.1a) or the Fredholm theory. In practice, an objective or subjective standard is set for what constitutes an "acceptable" (approximate) solution, and one of the following situations is observed:

(6.18)
- (i) An acceptable solution is obtained,
  or
- (ii) either no solution at all is found, or the computed solution is unacceptable.

In the *computationally singular* case (6.18ii), the method used to solve (1.1) or invert A may break down because A does not have a bounded inverse, or is analytically close to an operator $B \notin \mathcal{J}$. On the other hand, the algorithm employed may actually be trying to solve the system (5.44) with $\delta = 0$ and without (6.15) holding to the desired degree of accuracy. This will be called an *algebraic catastrophe of type I*. In the second situation described in (6.18ii), the acceptable particular

solution $\hat{w}$ may be contaminated by a complementary vector (6.17) to the extend that the resulting solution is unacceptable. This *algebraic catastrophe of type II* can occur in the numerical solution of differential equations by the use of approximating difference equations. For example, the difference equation

(6.19) $$3u_{n+1} + 8u_n - 3u_{n-1} = 0$$

with the initial conditions

(6.20) $$u_0 = 1, \quad u_1 = \frac{1}{3} ,$$

has the bounded solutions

(6.21) $$u_n = (\frac{1}{3})^n , \qquad n = 0,1,2,\ldots,$$

which may be the ones considered to be acceptable. However, a slight perturbation of (6.20), such as rounding $\frac{1}{3}$ to eight decimal places,

(6.22) $$w_0 = 1, \quad w_1 = 0.33333333$$

gives the corresponding solutions w_n of $3w_{n+1} + 8w_n - 3w_{n-1} = 0$ as

(6.23) $$w_n = (0.999999999)(\frac{1}{3})^n + (0.000000001)(-3)^n ,$$

$n = 1,2,\ldots,$ and the second term on the right-hand side of (6.23) will eventually wreak havoc with the accuracy of the approximation of u_n by w_n.

As indicated in §1b, if the operator A is singular, then a generalized inverse $A^\dagger$ of A having certain useful properties may be sought, for example, to give a solution of (1.1) in the form (1.4) if (1.1) is consistent. As (1.5) indicates, the vector $x = A^\dagger y$ will be a particular solution of (1.1) for any inner inverse $A^\dagger$ of A. An algebraic perturbation method may be used to obtain inner inverses of singular operators which have a Fredholm theory, under the technical assumption that the space Y is reflexive, that is, $Y^{**} = Y$ [38, p.192]. In this case, if

(6.24) $$U^* = \{u_1^*, u_2^*, \ldots, u_d^*\} \subset Y^*$$

is a set of linearly independent functionals on Y, then the Hahn-Banach theorem guarantees the existence of a set of d linearly independent vectors in Y to which the Gram-Schmidt orthonormalization process [38, p. 116] may be applied, if necessary, to obtain the set

(6.25) $$U = \{u_1, u_2, \ldots, u_d\} \subset Y$$

for which

(6.26) $$\langle u_i^*, u_j \rangle = \langle u_i, u_j^* \rangle = \delta_{ij} , \qquad i,j = 1,2,\ldots,d ,$$

where δ_{ij} again denotes the Kronecker delta. Similarly, given a set of linearly independent vectors

(6.27) $$V = \{v_1, v_2, \ldots, v_d\} \subset X,$$

a set of functionals

(6.28) $$V^* = \{v_1^*, v_2^*, \ldots, v_d^*\} \subset X^*$$

exists such that

(6.29) $$\langle v_i^*, v_j \rangle = \delta_{ij}, \qquad i,j = 1,2,\ldots,d,$$

whether X is reflexive or not.

Theorem 6.2. Suppose that $A \in L(X,Y)$ has a Fredholm theory, and

(6.30) $$u^* A = Av = 0$$

if and only if $u^* \in span\{u_1^*, u_2^*, \ldots, u_d^*\} \subset Y^*$ and $v \in span\{v_1, v_2, \ldots, v_d\} \subset X$, where the *defect* d of A is positive. Then, for $u_k \in U$ and $v_k^* \in V^*$, $k = 1,2,\ldots,d$, where U and V^* are defined by (6.24)-(6.29), the operator

(6.31) $$B = A - \sum_{k=1}^{d} u_k \rangle\langle v_k^*$$

is invertible, and

(6.32) $$AB^{-1}A = A,$$

so that $A^\dagger = B^{-1}$ is an inner inverse of A.

Proof: To show that B is invertible, consider the homogeneous equation $Bz = 0$, which is equivalent to

(6.33) $$Az = \sum_{k=1}^{d} u_k \langle v_k^*, z \rangle .$$

As this equation is solvable if and only if the right-hand side is orthogonal to $u_1^*, u_2^*, \ldots, u_d^*$ because A has a Fredholm theory, it follows from (6.26) that

(6.34) $$\langle v_k^*, z \rangle = 0, \quad k = 1,2,\ldots,d,$$

and thus $Az = 0$. This means that z is of the form $z = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_d v_d$, where the coefficients α_k are given by (6.34), and hence $z = 0$ is the unique solution of the homogeneous equation $Bz = 0$, which implies the existence of B^{-1}. To prove (6.32), note that from (6.29), (6.30), and (6.31),

(6.35) $$Bv_i = -\sum_{k=1}^{d} u_k \langle v_k^*, v_i \rangle = -u_i, \qquad i = 1,2,\ldots,d,$$

hence

(6.36) $$B^{-1}u_k = -v_k, \qquad k = 1,2,\ldots,d,$$

and

(6.37) $$B^{-1}A = B^{-1}\left(B + \sum_{k=1}^{d} u_k \rangle\langle v_k^*\right) = I - \sum_{k=1}^{n} v_k \rangle\langle v_k^*,$$

and (6.32) follows directly from (6.30). QED

Instead of (6.36), one could also use the relationships

(6.38) $$v_k^* B^{-1} = -u_k^*, \qquad k = 1,2,\ldots,d,$$

to establish (6.32). The operator $B^{-1} = A^\dagger$ obtained from (6.31) is called *Hurwitz pseudoinverse* of A [31, pp. 165-168; 12], which goes back to 1912.

By the same reasoning as above, any operator of the form

(6.39) $$A^\dagger = \left(A - \sum_{k=1}^{d} u_k \rangle \beta_k \langle v_k^*\right)^{-1}$$

for $\beta_1, \beta_2, \ldots, \beta_d$ such that $\beta_1\beta_2\ldots\beta_d \neq 0$ will be an inner inverse of A. However, as these operators are invertible, they cannot satisfy condition (2) of §1b which characterizes outer inverses; consequently, the construction (6.39), while useful for some purposes, only gives a partial solution to the problem of finding generalized inverses.

Another matter of computational importance relates to the calculation of generalized inverses of perturbations of operators with known generalized inverses. Suppose, for example, that one has an efficient technique to obtain the Moore-Penrose generalized inverse $A^\dagger$ of A [27], and then would like to use the result to obtain the generalized inverses of perturbed operators $B = A + \Delta A$ with less effort than calculating $B^\dagger$ *ab initio*, or error bounds for the approximation of $B^\dagger$ by $A^\dagger$. As $A^\dagger$ is not a continuous function of A in general, it would be expected that analytic perturbation methods apply only under restrictive conditions, as even for $\|\Delta A\|$ arbitrarily small, one of the algebraic catastrophes that the rank of B is greater or less than the rank of A could occur. Most applications of analytic perturbation theory to the above problems are carried out under assumptions that ensure $rank(B) = rank(A)$, or that the change in rank is known [23, pp. 333-351]. Algebraic perturbation methods, on the other hand, are not necessarily subject to this kind of limitation. For rank one modifications of A, C. D. Meyer, Jr. [19; 23, pp. 351-352] has obtained formulas of the type

$$(A + u \gtrless v)^\dagger = A^\dagger + G \tag{6.40}$$

for all six possible cases, where G depends on $A^\dagger$ and the data. More general finite-rank modifications (5.42) of A can then be handled by the method of successive rank one modifications corresponding to (5.51)-(5.52). This latter algorithm was originated by Greville [9] for the recursive calculation of the Moore-Penrose generalized inverse of a matrix. Formula (6.40) reduces to (5.28) in the special case that A is invertible, as for any generalized inverse of A, $A^\dagger = A^{-1}$ for all $A \in \mathcal{I}$. This suggests the computational strategy of using a method for generalized inversion on an operator which is suspected of being singular or nearly singular. If the operator or the perturbed operator actually involved in the calculation is nonsingular, then this technique will yield its inverse, whereas a straightforward inversion method might fail.

Another approach to ill-posed problems is to approximate them by a perturbed problem which is well conditioned. An example is the technique of *regularization*, due to A. N. Tihonov [39, 40], which has close connections with the subject of generalized inverses [22]. If the operator A in (1.1) does not have a bounded inverse, then the smallest perturbation Δy in the data can cause an enormous change Δx in

the solution of the perturbed problem (2.3) as compared to the solution of the reference problem. A typical situation in which problems of this type arise in applications is that X and Y are Hilbert spaces, and $A = K$ is a compact operator. The prototype of the resulting equation

(6.41) $$Kx = y\ ,\quad K \in \mathcal{K}\ ,$$

is the linear Fredholm integral equation of the *first kind* ,

(6.42) $$\int_0^1 K(s,t)x(t)dt = y(s), \qquad 0 \leq s \leq 1\ .$$

As perturbations in (6.42) in actual practice are inevitable, due to errors of measurement, discretization, and computation, direct numerical solution of (6.42) by standard techniques that work well for the integral equation (4.5) of second kind are rarely successful. The same observation may be made for (6.41) as compared to

(6.43) $$(\alpha I - K)x = y$$

for $\alpha \neq 0$. In order to find an acceptable approximate solution of the perturbed version of (6.41), the method of regularization consists of finding an element $w(\alpha) \in X$ which minimizes the functional

(6.44) $$f(w;\alpha) = \| Kw - z\|^2 + \alpha^2 \| w\|^2\ .$$

Thus, (6.44) represents a trade-off between the fidelity with which the perturbed equation $Kw = z$ is satisfied, and the size of the norm of the corresponding solution. The parameter α (or sometimes α^2) in (6.44) is called the *regularization parameter* . The crucial problem in this field is the determination of the *optimal* regularization parameter, for which the value of $f(w;\alpha)$ is minimum, or at least a method for obtaining good approximations to the optimal value. A significant recent advance in this area is the application by Grace Wahba [41] of the method of weighted cross-validation to the case that the perturbation is due to discretization of the data with random errors of the type known as "white noise".

7. The eigenvalue-eigenvector problem. As stated in §1c, this problem is to find eigenvalues λ and *right eigenvectors* $x \neq 0$ satisfying (1.9), where $A \in L(X,X)$, X a Hilbert space. It follows that one is interested in the values of λ for which the linear operator

(7.1) $$T(\lambda) = A - \lambda I$$

is singular, and one may also want to find the *left eigenvectors* $y \neq 0$ of A corresponding to the eigenvalue λ which satisfy the homogeneous equation

(7.2) $$y(A - \lambda I) = 0\ .$$

The additional assumption will be made that the values of λ considered are restricted to those for which $T(\lambda)$ has a Fredholm theory. This condition does not exclude any λ in the finite-dimensional algebraic case; however, for Fredholm integral operators of the first kind or compact operators in general, it is customary

to formulate the eigenvalue-eigenvector problem in terms of the reciprocal eigenvalue $\mu = 1/\lambda$, as the operator

$$S(\mu) = I - \mu K, \quad K \in \mathcal{K}, \tag{7.3}$$

will have a Fredholm theory for all scalars $\mu \in \Lambda$ by Theorem 5.3. This is equivalent to excluding $\lambda = 0$ from consideration in (7.1) if A is compact.

In order to contemplate the application of analytic perturbation methods to the eigenvalue-eigenvector problem, it is essential to determine conditions under which this problem is well-posed, as the operator $T(\lambda)$ will be singular if λ is an eigenvalue. One way to do this is to convert equation (1.9) and the normalization condition (1.10) into the nonlinear system

$$P(q) := \begin{pmatrix} Ax - \lambda x \\ \\ \frac{1}{2} - \frac{1}{2}\langle x, x \rangle \end{pmatrix} = 0 \tag{7.4}$$

in the product space $Q = X \times \Lambda$ of vectors $q = (x,\lambda)^T$, $x \in X$, $\lambda \in \Lambda$. Suppose that $q_1 = (x_1,\lambda_1)^T$ is a solution of (7.4); that is, λ_1 is an eigenvalue of A, and x_1 is a corresponding normalized eigenvector. Then, the implicit function theorem [10] guarantees continuous dependence of the solution of (7.4) on the data if the linear operator $P'(q_1) \in L(Q,Q)$ has a bounded inverse, where $P'(q)$ is the *Fréchet derivative*

$$P'(q) = \begin{pmatrix} A - \lambda I & -x \rangle \\ \\ -\langle x & 0 \end{pmatrix} \tag{7.5}$$

of the operator P at q [30, pp. 97-100]. The formulation (7.4), while not the most general [1], has the advantage that if A is *Hermitian* ($A^* = A$ [38, pp. 324-327]), then so is $P'(q)$. The following theorem gives an explicit formulation of the inverse operator $[P'(q_1)]^{-1}$ in this case if the defect of $T(\lambda)$ is equal to one, that is, if all solutions x of the homogeneous equation $T(\lambda_1)x = 0$ are scalar multiples of the normalized eigenvector x_1, making use of the fact that the right and left eigenvectors of an Hermitian operator can be identified.

Theorem 7.1. If A is Hermitian, $q_1 = (x_1,\lambda_1)^T$ satisfies (7.4), and the defect of $T(\lambda_1)$ is equal to one, then

$$[P'(q_1)]^{-1} = \begin{pmatrix} B_1^{-1} - x_1 \rangle\langle x_1 & -x_1 \rangle \\ \\ -\langle x_1 & 0 \end{pmatrix} \tag{7.6}$$

where

$$B_1^{-1} = (A - \lambda_1 I - x_1 \rangle\langle x_1)^{-1} \tag{7.7}$$

is the Hurwitz pseudoinverse of $A - \lambda_1 I$.

Proof: It follows by direct calculation and the use of (6.36) and (6.38) that

$$(7.8)\quad P'(q_1)\begin{pmatrix} B_1^{-1} - x_1 \rangle\langle x_1 & -x_1 \rangle \\ -\langle x_1 & 0 \end{pmatrix} = \begin{pmatrix} B_1^{-1} - x_1 \rangle\langle x_1 & -x_1 \rangle \\ -\langle x_1 & 0 \end{pmatrix} P'(q_1) =$$

$$= \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix},$$

the identity operator in $Q = X \times \Lambda$. QED

By the use of Theorem 6.2, formula (7.6) can be extended immediately to the non-Hermitian case $y_1 T(\lambda_1) = T(\lambda_1)x_1 = 0$, provided the defect of $T(\lambda_1)$ remains equal to one [1, §3]. Under these circumstances, results are available by the methods of analytic perturbation theory similar to those for nonsingular linear equations (1.1) [1, §5].

For the finite-dimensional case, perturbation methods and error analysis for the algebraic eigenvalue problem have been presented in great detail in the comprehensive work by J. H. Wilkinson [44, pp. 62-188]. Just one of these results will be cited here, which fits into the framework of algebraic perturbation theory. Suppose that w is a unit vector, and $p = (w,\mu)^T$ is an approximate solution of (7.4), so that

$$(7.9)\qquad (A - \mu I)w = r,$$

with residual vector r. From equation (5.38), it follows that

$$(7.10)\qquad (A - r\rangle\langle w^* - \mu I)w = 0,$$

so that w is an exact eigenvector of the perturbed operator

$$(7.11)\qquad B = A - r\rangle\langle w^*$$

corresponding to the eigenvalue μ [44, pp. 170-171]. The perturbed operator B is simply a rank one modification of the reference operator A.

Another application of algebraic perturbation theory to the eigenvalue-eigenvector problem has been given by W. Stenger [37] to find inequalities between eigenvalues of perturbed and reference integral operators.

8. Linear programming. The solution of linear programming problems as formulated in §1d is one of the primary tools for decision making in government and commerce at the present time [8]. The number of variables involved is typically large, and a lot of computer time is expended for this purpose. Thus, an application of perturbation theory which would increase efficiency could result in substantial savings. Once again, the fact that the solutions do not depend continuously on the data in general limits the applicability of analytic perturbation techniques. A necessary and

sufficient condition for continuous dependence of the solution of the primal and dual linear programming problems in a neighborhood of solvable reference problems has been given recently by S. M. Robinson [34]. Studies of what is called *parametric programming* give conditions under which the solution of the reference problem remains unchanged under perturbation of the data [8, pp. 144-154]. On the subject of error estimation, P. Wolfe [45] has contributed a method for error analysis and control in the solution of linear programming problems.

Although changes in the objective function (1.14) are not usually difficult to deal with, perturbations in the constraints (1.15), as would result, for example, by the introduction of a new technology in an industry, may require the complete restarting of the solution method used. Consequently, the following problem may be of practical interest.

Problem 8.1. Given the solution x of (1.14)-(1.15) and the associated information, such as the choice of pivots in the simplex algorithm [45], find an efficient method for solving

(8.1) $$\textit{minimize } f(w) := \langle d, w \rangle + \eta$$

subject to

(8.2) $$B w \leq z , \quad w \geq 0 ,$$

where all perturbations in the reference data are of finite rank which is small compared to the size of the reference problem.

9. Nonlinear problems. Although this survey has been concerned mainly with linear problems, it should be mentioned that perturbation methods are widely applied to the solution of nonlinear operator equations

(9.1) $$P(x) = 0 ,$$

where P maps X into Y, and also *fixed point problems* in X of the form

(9.2) $$x = H(x) .$$

(It is evident that (9.2) is a special case of (9.1); conversely, there are many ways to convert (9.1) into an equivalent fixed point problem.)

These problems are well-posed in the neighborhood of a solution x_0 if, for example, H is continuous and contractive [30, Chapter 2], or, more restrictively, if P is differentiable and

(9.3) $$\Gamma_0 = [P'(x_0)]^{-1} \in L(Y,X).$$

Depending on the smoothness of P, in this case one can base analytic perturbation techniques on the implicit function theorem [10], Newton's method and its variants, Taylor series expansions, inversion of power series, and so on [30, Chapter 4]. These methods are all essentially derived from the corresponding ideas of elementary scalar calculus.

Recently, W. Rheinboldt has given generalizations of the condition numbers (5.8) and (5.9) for nonlinear operators for which (9.3) holds, and a corresponding generalization of the perturbation formula (5.13) for error estimation [33].

Algebraic perturbation methods for nonlinear operator equations are less well investigated. A nonlinear operator F with range belonging to the finite-dimensional space

$$Y_n = \mathit{span}\,\{y_1, y_2, \ldots, y_n\} \tag{9.4}$$

will be of the form

$$F(\cdot) = \sum_{j=1}^{n} y_j > f_j(\cdot) , \tag{9.5}$$

where $f_1(\cdot), f_2(\cdot), \ldots, f_n(\cdot)$ are (generally nonlinear) functionals on X. The perturbed operator equation

$$Q(x) = 0 , \tag{9.6}$$

where $Q = P - F$, is equivalent to the equation

$$P(x) = \sum_{j=1}^{n} \xi_j y_j , \tag{9.7}$$

where

$$\xi_j = f_j(x) , \quad j = 1,2,\ldots,n . \tag{9.8}$$

Suppose, and this is the *big assumption*, that the equation $P(x) = y$ is solvable for $y \in Y_n$, that is, an operator G is known which gives

$$x = G(\xi_1, \xi_2, \ldots, \xi_n) \tag{9.9}$$

if $P(x) = y$ is of the form (9.7). Then, applying $f_1, f_2, \ldots, f_n$ in turn to (9.9) yields the nonlinear system

$$\xi_i = h_i(\xi_1, \xi_2, \ldots, \xi_n) , \quad i = 1,2,\ldots,n , \tag{9.10}$$

where $h_1 = f_1 G$, $h_2 = f_2 G, \ldots, h_n = f_n G$, which is a finite-dimensional fixed-point problem in Λ^n of the form (9.2). On the basis of the *additional assumption* that (9.10) is solvable, the substitution of its solutions $\xi_1, \xi_2, \ldots, \xi_n$ into (9.9) provides a solution x of the nonlinear operator equation (9.6). As an example of this approach, the *Hammerstein integral equation* with kernel (5.31)

$$x(s) - \int_0^1 K(s,t)\phi(t,x(t))dt = 0 \tag{9.11}$$

is a rank one modification of the *nonlinear Volterra integral equation*

$$x(s) - \int_0^s L(s,t)\phi(t,x(t))dt = 0 \tag{9.12}$$

with kernel (5.33). Thus, if one can solve

$$x(s) - \int_0^s L(s,t)\phi(t,x(t))dt = \xi u(s) , \tag{9.13}$$

where

$$\xi = \int_0^1 v(t)\phi(t,x(t))dt . \tag{9.14}$$

for $x(s) = g(s;\xi)$, then from (9.14), the system (9.10) is equivalent to the scalar fixed point problem

$$\xi = h(\xi) := \int_0^1 v(t)\phi(t,g(t;\xi))dt \,, \tag{9.15}$$

which is one nonlinear equation in one unknown [32, §5].

Although quite a bit is known about nonlinear systems (9.10) in finite-dimensional spaces [28], the theory and practice of their solution is far from the highly developed technology available for finite linear systems (5.44). There is also the ever-present big assumption. Even though (9.9) is not obtainable explicitly, the form of the problem (9.7) suggests iteration: Solve (9.7) for given $\xi_1^{(0)}, \xi_2^{(0)}, \ldots, \xi_n^{(0)}$, substitute into (9.10) to obtain

$$\xi_i^{(1)} = h_i(\xi_1^{(0)}, \xi_2^{(0)}, \ldots, \xi_n^{(0)}), \quad i = 1,2,\ldots,n \,, \tag{9.16}$$

and so on. In the case that (9.6) is a boundary-value problem for a nonlinear differential equation, this is called "shooting" [13, Chapter 2, also §6.1]. Of course, this iteration may not converge, and some other method for solving (9.6) may be more appropriate.

This section will also conclude with an important problem, as much more work needs to be done.

Problem 9.1. For differentiable P, develop existence theory and find effective techniques for computing solutions x_0 of the nonlinear operator equation (9.1) in the case that $P'(x_0)$ does not have a bounded inverse.

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Linear problems of central interest in numerical analysis are the solution of linear equations, the construction of the inverse or a generalized inverse of a linear operator, finding the eigenvalues and eigenvectors of a linear operator, and linear programming. A survey is made of methods which apply if the data of a solved linear problem is perturbed by operators and vectors of small norm (analytic perturbation), or by operators of finite rank and vectors belonging to a finite-dimensional subspace (algebraic perturbation). Perturbation methods may be used to extend the theory of linear problems, to obtain economy of effort in the solution of perturbed problems, and to estimate errors due to inaccurate data and computation.

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